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New BPS solitons in 2+1 dimensional noncommutative CP^1 model

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Abstract: Investigating the solitons in the non-commutative CP^1 model, we have found a new set of BPS solitons which does not have counterparts in the commutative model.

Keywords: Sigma Models, Solitons Monopoles and Instantons, Non-Commutative Geometry.

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Contents

1. Introduction

Field theories on the non-commutative space have been extensively studied in the last few years. Particularly, BPS solitons are interesting, because they might not share the common features with those on the commutative space [1, 2, 3].

Solitons in the non-commutative CP^1 model have been studied in [4] and further developed in [5] in connection with the dynamical aspects of the theory. The non-BPS solitons, that do not exist in the commutative case, have been studied in [6]. These investigations were reviewed in [7].

In this paper, we report on a set of new BPS solitons in the non-commutative CP^1 model, that does not exist in the commutative limit.

We consider the CP^1 model on 2+1 dimensional non-commutative spacetime. The space coordinates obey the commutation relation

$$[x, y] = i\theta \,, \tag{1.1}$$

or

$$[z, \bar{z}] = \theta > 0 \,, \tag{1.2}$$

in terms of the complex variables, $z = \frac{1}{\sqrt{2}}(x + iy)$ and $\bar{z} = \frac{1}{\sqrt{2}}(x - iy)$. The Hilbert space can be described in terms of the energy eigenstates $|n\rangle$ of the harmonic oscillator whose creation and annihilation operators are $\bar{z}$ and z respectively,

$$\begin{aligned} z\,|n\rangle &= \sqrt{\theta n}\,|n-1\rangle \,, \\ \bar{z}\,|n\rangle &= \sqrt{\theta(n+1)}\,|n+1\rangle \,, \end{aligned} \tag{1.3}$$

The CP^1 lagrangian is

$$L = \mathrm{Tr}\left(|D_t\Phi|^2 - |D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2\right), \tag{1.4}$$

where Φ is a 2-component complex vector with the constraint $\Phi^\dagger\Phi = 1$. We consider Φ to be the fundamental field and thus to be non-singular. Tr denotes the trace over the Hilbert space as

$$\mathrm{Tr}\,\mathcal{O} = 2\pi\theta \sum_{n=0}^{\infty} \langle n|\,\mathcal{O}\,|n\rangle \,. \tag{1.5}$$

The covariant derivative is defined by

$$D_a\Phi = \partial_a\Phi - i\Phi A_a\,,\quad A_a = -i\Phi^\dagger\partial_a\Phi\,,\qquad (a = t, z, \bar{z})\,,\tag{1.6}$$

where $\partial_z = -\theta^{-1}[\bar{z},\ ]$ and $\partial_{\bar{z}} = \theta^{-1}[z,\ ]$.

For the static configurations, topological charge and static energy are given by

$$Q = \frac{1}{2\pi}\,\mathrm{Tr}\left(|D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2\right),\tag{1.7}$$

and

$$E = \mathrm{Tr}\left(|D_z\Phi|^2 + |D_{\bar{z}}\Phi|^2\right) \geq 2\pi\,|Q|\,.\tag{1.8}$$

The configuration which saturates the energy bound satisfies the BPS soliton equation

$$D_{\bar{z}}\Phi = 0\,,\tag{1.9}$$

or BPS anti-soliton equation

$$D_z\Phi = 0\,.\tag{1.10}$$

It is convenient [4] to introduce the 2-component complex vector W and the projection operator P as

$$\Phi = W\frac{1}{\sqrt{W^\dagger W}}\,,\qquad P = \Phi\Phi^\dagger\,.\tag{1.11}$$

In terms of the projection operator, BPS soliton equations (1.9) and (1.10) are [8, 9, 10]

$$(1-P)\,zP = 0\,,\tag{1.12}$$

and

$$(1-P)\,\bar{z}P = 0\,,\tag{1.13}$$

which indicate respectively

$$zW = WV\,,\tag{1.14}$$

and

$$\bar{z}W = WV\,,\tag{1.15}$$

where V is a scalar function [4]. Topological charge (1.7) and static energy (1.8) can be expressed as

$$Q = \frac{1}{2\pi}\,\mathrm{Tr}\left\{\partial_{\bar{z}}\Phi^\dagger\,(1-P)\,\partial_z\Phi - \partial_z\Phi^\dagger\,(1-P)\,\partial_{\bar{z}}\Phi\right\},\tag{1.16}$$

and

$$E = \mathrm{Tr}\left\{\partial_{\bar{z}}\Phi^\dagger\,(1-P)\,\partial_z\Phi + \partial_z\Phi^\dagger\,(1-P)\,\partial_{\bar{z}}\Phi\right\}.\tag{1.17}$$

The examples of BPS soliton are $W = (z^n, 1)^t$, with topological charge $Q = n$ and energy $E = 2\pi n$, and those of BPS anti-soliton are $W = (\bar{z}^n, 1)^t$ with topological charge $Q = -n$ and energy $E = 2\pi n$ [4]. These configurations are solitons also in the commutative theory.

2. New solitons

We have found that the following is the BPS soliton solution of the non-commutative CP^1 model. The configuration of the soliton with the topological charge $Q = -n$ is

$$\Phi = \begin{pmatrix} \bar{z}^n \dfrac{1}{\sqrt{\prod_{l=1}^{n}(\bar{z}z + l\theta)}} \\ 0 \end{pmatrix}, \tag{2.1}$$

which can also be expressed in terms of projection operator P as

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.2}$$

For $Q = n$, on the other hand, the soliton can be written as

$$\Phi = \begin{pmatrix} \dfrac{1}{\sqrt{\prod_{l=1}^{n}(\bar{z}z + l\theta)}} z^n \\ \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix}, \tag{2.3}$$

and the corresponding projection operator expression is

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix}. \tag{2.4}$$

We can straightforwardly confirm that (2.2) and (2.4) satisfy the BPS equations (1.13) and (1.12) respectively. The energy of these solitons are of course $E = 2\pi n$ due to the BPS property.

We calculate the topological charge of (2.1) and (2.3). For BPS anti-solitons (2.1), we use

$$\partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_z \Phi = 0 \tag{2.5}$$

which follows from (1.10), and

$$\begin{aligned} \partial_z \Phi^\dagger (1 - P) \partial_{\bar{z}} \Phi &= \theta^{-2} \left[\frac{1}{\sqrt{\prod_{l=1}^{n}(\bar{z}z + l\theta)}} z^n, \bar{z} \right] \left(\sum_{m=0}^{n-1} |m\rangle \langle m| \right) \left[z, \bar{z}^n \frac{1}{\sqrt{\prod_{l=1}^{n}(\bar{z}z + l\theta)}} \right] \\ &= \theta^{-1} n |0\rangle \langle 0| \,. \end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) into (1.16), the topological charge is

$$Q = \frac{1}{2\pi} 2\pi\theta \sum_{k=0}^{\infty} \langle k| \left(-\theta^{-1} n |0\rangle \langle 0| \right) |k\rangle = -n \,. \tag{2.7}$$

Similarly, for BPS solitons (2.3), we use

$$\partial_z \Phi^\dagger (1 - P) \partial_{\bar{z}} \Phi = 0 \tag{2.8}$$

which follows from (1.9), and

$$\begin{aligned} \partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_z \Phi &= \theta^{-2} \left[\sum_{l=0}^{n-1} |l\rangle \langle l| \,, z \right] \left(1 - \sum_{m=0}^{n-1} |m\rangle \langle m| \right) \left[\bar{z}, \sum_{l=0}^{n-1} |l\rangle \langle l| \right] \\ &= \theta^{-1} n |n-1\rangle \langle n-1| \,. \end{aligned} \tag{2.9}$$

The topological charge is $Q = n$.

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3. Discussions

First we note that these solutions do not have the commutative counterparts. In order to see this, we consider

$$W = \begin{pmatrix} a^{-n}\bar{z}^n \prod_{l=1}^{n} (\bar{z}z + l\theta)^{-1} \\ 1 \end{pmatrix} \tag{3.1}$$

for $Q < 0$ and

$$W = \begin{pmatrix} a^{-n} \left(\prod_{l=1}^{n} (\bar{z}z + l\theta)^{-1}\right) z^n \\ 1 \end{pmatrix} \tag{3.2}$$

for $Q > 0$, where a is a real parameter. Taking the limit $a \to 0$ in the non-commutative case, Φ reduces to the non-singular configurations (2.1) and (2.3) respectively. On the other hand, in the commutative case, the same limit leads to

$$\Phi = \begin{pmatrix} e^{-in\varphi} \\ 0 \end{pmatrix} \tag{3.3}$$

and

$$\Phi = \begin{pmatrix} e^{in\varphi} \\ 0 \end{pmatrix}, \tag{3.4}$$

for $z \neq 0$, where $z \equiv |z|\, e^{i\varphi}$. For $z = 0$, the limit leads to

$$\Phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5}$$

in both cases. There exists a discontinuous jump at the origin. Which shows that in the commutative theory the configurations of CP^1 field Φ corresponding to $a \to 0$ limit of (3.1), (3.2) are absent. Thus, we have seen that the singular configurations in the commutative theory become non-singular in the non-commutative case which can be attributed to non-commutativity of the coordinates.

A relation to the U(2) sigma model discussed in ref. [9] is interesting. If we define $U = 1 - 2P$ using the projection operator P of (1.11), U satisfies $U^\dagger U = U^2 = 1$ and as such is a U(2) field. Furthermore, the BPS equation in U(2) sigma model are expressed as (1.12) and (1.13) [9]. From this viewpoint, our solutions (2.2) and (2.4) could be considered as embedding of abelian solutions into U(2) sigma model. Actually, our projector P for CP^1 solitons is unitarily equivalent to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} P_n & 0 \\ 0 & 0 \end{pmatrix} \tag{3.6}$$

for $Q = -n$ and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P_n \end{pmatrix}, \tag{3.7}$$

for $Q = n$, where P_n is the projector for abelian solitons of ref. [9]. Accordingly, the computations of energy for our solitons could be reduced to those for the embedded abelian solitons. Which makes it easy to verify $E = 2\pi n$ by means of BPS equations.

Finally, a few words on the properties of the solutions (2.1) and (2.3). It follows from (1.11), that with the scalar function Δ that commutes with $\bar{W}W$, the transformation $W \to W\Delta$ does not change Φ. In order to take the $a \to 0$ limit in (3.1) (3.2), we can use this invariance and rewrite these as

$$W = \begin{pmatrix} \bar{z}^n \\ a^n \prod_{l=1}^n (\bar{z}z + l\theta) \end{pmatrix}, \tag{3.8}$$

and

$$W = \begin{pmatrix} z^n \\ a^n \bar{z}^n z^n + \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix}, \tag{3.9}$$

where

$$\Delta = a^n \prod_{l=1}^{n} (\bar{z}z + l\theta) \tag{3.10}$$

for (3.1) and

$$\Delta = a^n \bar{z}^n z^n + \sum_{m=0}^{n-1} |m\rangle \langle m| \tag{3.11}$$

for (3.2). Taking the limit of $a \to 0$, the BPS soliton solutions (3.1) (3.2) can be expressed in terms of W as

$$W = \begin{pmatrix} \bar{z}^n \\ 0 \end{pmatrix}, \tag{3.12}$$

and

$$W = \begin{pmatrix} z^n \\ \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix}. \tag{3.13}$$

The configurations (3.12) and (3.13) satisfy the BPS equations (1.15) and (1.14) with

$$V = \bar{z} \tag{3.14}$$

and

$$V = z - \sqrt{n\theta}\, |n-1\rangle \langle n| , \tag{3.15}$$

respectively.

To summarize, we have considered solitons in the non-commutative CP^1 model and have found new solitons that do not have their counterparts in the commutative theory. Further properties of these solutions will be examined using more general approaches [11, 12, 13].

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